A SPECTRAL UNCERTAINTY QUANTIFICATION METHOD TO TRANSOCEANIC TSUNAMI PROPAGATION

Gregorio Pellegrini
University of Verona, Italy, *gregorio.pellegrini@gmail.com*

ABSTRACT

Uncertainty Quantification (UQ) methods investigate the effects of uncertainties in simulations, improving the validation of the numerical method, providing a finest prediction of an empirical phenomenon.

A tsunami propagation is analyzed setting two physical parameters as uncertain, then the UQ-results are compared with empirical measurements.

INTRODUCTION

Let $Y = \mathcal{M}(X, \Theta)$ be a 1D-process, where X is the random parameter, while Θ gathers the deterministic ones.
The approximation of Y is ruled by the so-called **basic random variable** ξ: a continuous random variable such that $X = T(\xi)$ for a suitable transformation T (*Inverse transform method*), thus $\mathcal{M}$ is expressed in terms of ξ, namely $\overline{\mathcal{M}} = \mathcal{M} \circ T$.

The univariate **Polynomial Chaos Expansion (PCE)** of $Y = \overline{\mathcal{M}}(\xi, \Theta)$ is defined in the Hilbert space $\left(L^2(\Omega, \sigma(\xi), \mathbb{P}), \langle\cdot,\cdot\rangle_{\mathbb{P}}\right)$ as

$$Y^{(N)} = \sum_{i=0}^{N} c_i \Psi_i(\xi) \, , c_i = \frac{\langle \overline{\mathcal{M}}, \Psi_i \rangle_{\mathbb{P}}}{\|\Psi_i\|^2_{\mathbb{P}}} \, , \qquad (1)$$

where N is the degree of truncation and $\{\Psi_i(\xi)\}_{i=0}^N$ are the first N elements of the so-called *generalized polynomial Chaos* basis **(gPC)**, which is a family of **orthogonal polynomials**, related to the choice of ξ

ξ	Normal r.v.	Uniform r.v.	Exponential r.v.
$\{\Psi_i\}_{i=0}^{+\infty}$	Hermite	Legendre	Laguerre

The PCE-coefficients are computed by means of **Non-Intrusive Spectral Projection (NISP)** method which exploits a suitable **Gaussian quadrature formula**:

$$c_i = \frac{\int_D \overbrace{\overline{\mathcal{M}}(x)\Psi_i(x)}^{g_i(x)} w(x)dx}{\|\Psi_i\|^2_{\mathbb{P}}} \approx \frac{\sum_{j=1}^{N_Q} g_i(x_j)w_j}{\|\Psi_i\|^2_{\mathbb{P}}}$$

where the quadrature nodes $\{x_j\}_{j=1}^{N_Q}$ read as a particular set of realizations of ξ.

The *gPC* basis for **L-variate PCE** is the *tensorization* of 1D-gPC bases, one for each independent entry of the random vector $\boldsymbol{\xi}(\omega) \in \mathbb{R}^L$. Then the multivariate PCE, labeled by the multi-index $\mathbf{i}$, is

$$Y^{(N)} = \sum_{|\mathbf{i}| \le N} c_{\mathbf{i}} \Psi_{\mathbf{i}}(\boldsymbol{\xi}) \, , \qquad (2)$$

which is truncated at **total degree** N, thus $\#\{\Psi_{\mathbf{i}}(\boldsymbol{\xi}) \; : \; |\mathbf{i}| \le N\} = \binom{N+L}{N}$. Then multivariate Gaussian quadrature formulas are used to detect the coefficients (NISP).

REFERENCES

[1] O.P. LE MAÎTRE, O.M. KNIO, *Spectral Methods for Uncertainty Quantification*, Springer, 2010

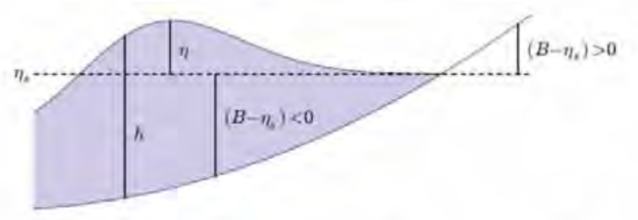
[2] R.J. LEVEQUE, D.L. GEORGE, M.J. BERGER. *Tsunami modeling with adaptively refined finite volume methods*, Acta Numerica, pp. 211-289, (2011).

TSUNAMI PROPAGATION

Following [2], the 2D-shallow water equations model the tsunami propagation

$$h_t + (hu)_x + (hv)_y = 0 \, ,$$
$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y = -ghB_x \, ,$$
$$(hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_y = -ghB_y \, ,$$

where $u(x, y, t)$ and $v(x, y, t)$ are the depth-averaged velocities and $B(x, y, t)$ is the surface elevation relative to mean sea level η_s.



The tailored **Geoclaw** software, based on **Finite Volume Method**, see [2], computes the numerical solution on the spatial domain D (longitude times latitude) and the time domain T (hours)

$$D = [-120, -60] \times [-60, 0] \, , \; T = [0, 9]$$

An uniform 240×240 rectangular grid discretizes D, such points are labeled by $\{\mathbf{x}_k\}_{k=1}^{240\times240}$, while time integration uses an adaptive time steps technique, thus the solution is evaluated at $\{t_h\}_{h=1}^M$.

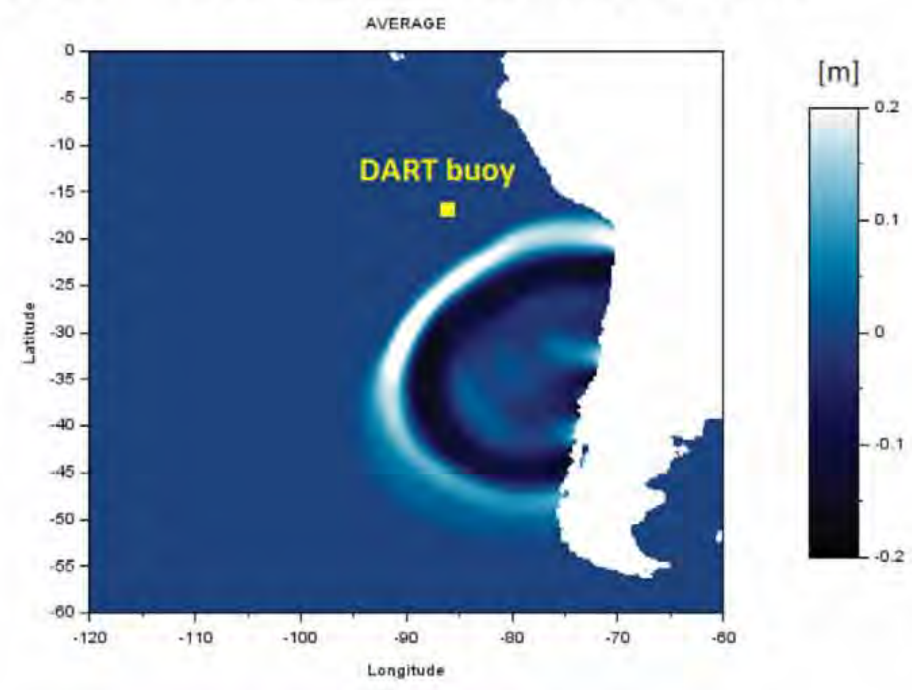
In order to embed in the model fluctuations of the **constant of gravity** g and the **Earth's ellipsoid shape** , we set $g \sim \mathcal{N}(9.795, 0.012)$ [m/s^2] and the radius $R \sim \mathcal{U}(6378.137, 6362.132)$ [Km] .
Then, for each couple $(t_h, \mathbf{x}_k)$ the two dimensional PCE-approximation of the **surface elevation** η is detected

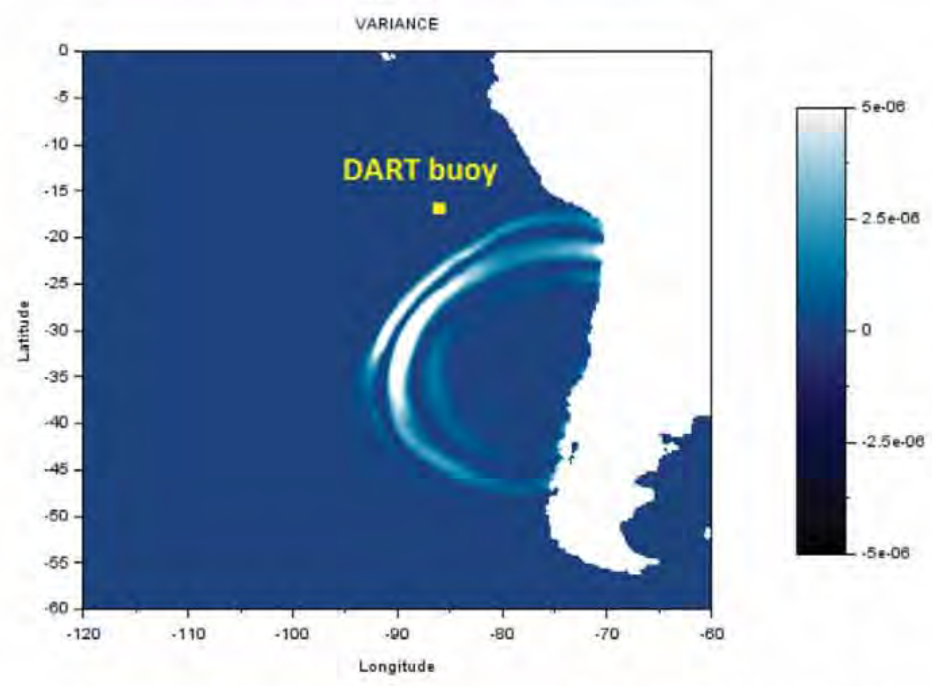
$$\eta^{(N)}(t_h, \mathbf{x}_k, \boldsymbol{\xi}) = \sum_{|\mathbf{i}| \le N} c_{\mathbf{i}}(t_h, \mathbf{x}_k)\Psi_{\mathbf{i}}(\boldsymbol{\xi}) \qquad (3)$$

where $N = 5$, $\mathbf{i}$ is a 2D multi-index and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, where the former represents the gravity, while the latter is the radius.
NISP toolbox for **Scilab** software implements NISP method. $\overline{\mathcal{M}}$ is the surface elevation numerically detected by Geoclaw. Thus few numerical solutions (one for each realization of $\boldsymbol{\xi}$) are required, providing a **not time consuming** method meanwhile preserving **high accuracy** of the approximation.

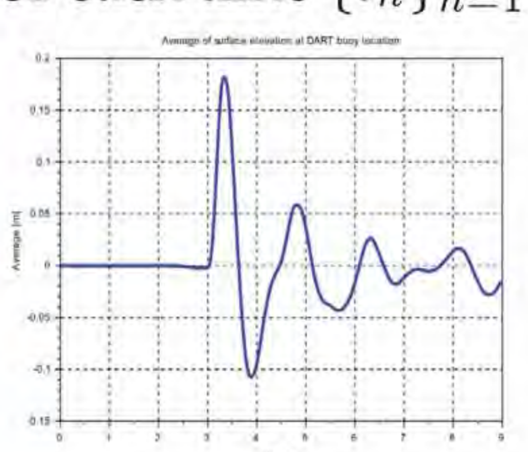
THE 27TH FEBRUARY 2010 CHILE TSUNAMI

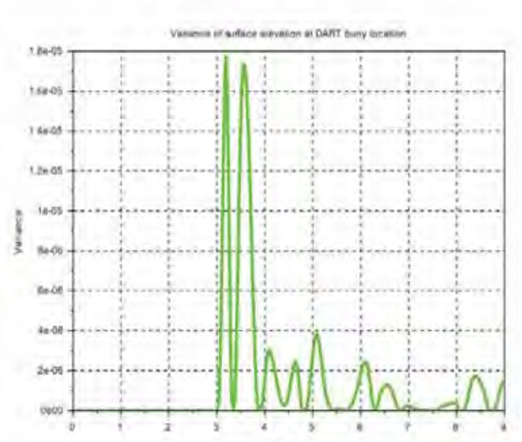
The mean and variance of the surface elevation at $t = 2.5$ [h] are shown below.



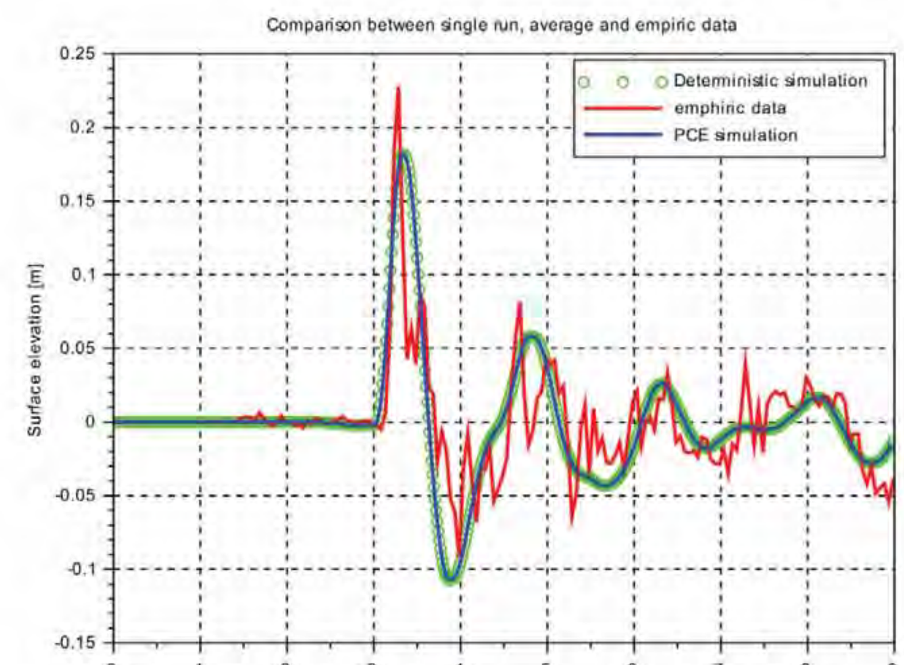


Let us focus on the surface elevation at **DART buoy** location ($17°58'48"S, 86°19'48"W$), displaying the mean and variance of PCE-approximation for each time $\{t_h\}_{h=1}^M$





Where zero values of the variance (green) appear, exact knowledge of η is available: the true value coincides with average (blue). Since UQ methods aim to improve **validation** of the simulation, let us compare the mean of (3) and a deterministic simulation of the surface elevation, run for the mean of g and R, with empirical surveys (one every 3 minutes) taken by the DART buoy.



In the above figure, both deterministic and PCE simulation fits empirical data, furthermore to achieve fair comparison let us compute the error with respect to empirical data.

Type of simulation	Error Infinity norm	Error ℓ_2 norm
PCE (UQ method)	0.1336395	0.3632628
Deterministic	0.1344455	0.3648620
% improvement moving to UQ	- 0.60 %	- 0.44 %

The **percentage variation** of the two errors (infinity norm and euclidean one) is computed, highlighting improvement in validation of PCE-approximation.

CONCLUSION

Although the magnitude of the variance for surface elevation at DART location is small, by embedding uncertain parameters in the model we accomplish better fitting to empirical data, providing more **reliable simulation** of the physical phenomenon.